# NEUTRAL STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS IN INFINITE-DIMENSIONAL SPACES

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We consider the neutral type stochastic functional-differential equations of the form

$$\begin{aligned} d\Big[u(t)+g(u_t)\Big] &= \Big[Au+f(u_t)\Big]dt+\sigma(u_t)dW(t) \qquad \text{for} \quad t>0; \\ u(t) &= \varphi(t), \qquad\qquad\qquad\qquad\qquad\qquad t\in[-h,0),\ h>0 \end{aligned} \tag{1}$$

Here A is an inifinitesimal generator of a strong continuous semigroup $\{S(t),\ t\geq 0\}$ of bounded linear operators in real separable Hilbert space H. The noise $W(t)$ is a $Q-$Wiener process on a separable Hilbert space K. For any $h>0$ denote $C_h := C([-h,0],H)$ to be a space of continuous H-valued functions $\varphi : [-h,0]\to H$, equipped with the norm

$$||\varphi||_{C_h} := \sup_{t\in[-h,0]} ||\varphi(t)||$$

where $||\cdot||_H$ stands for the norm in H. Throughout this talk will be denoted with $||\cdot||$. The solution $u(t)$ of (1) is sometimes referred as a **state process.**

We also denote $u_t := u(t+\theta)$, where $\theta \in [-h, 0)$. The functionals $f$ and $g$ map $C_h$ to $H$, and $\sigma : C_h \to L_2^0$, where $L_2^0 = L\Big(Q^{\frac{1}{2}}K, H\Big)$ is the space of Hilbert-Schmidt operators from $Q^{\frac{1}{2}}K$ to H.
Finally, $\phi : [-h, 0] \times \Omega \to H$ is the initial condition, where $(\Omega, F, P)$ is the probability space.

Differential equations of neutral type arise in modeling the memory effects in various dynamical processes (e.g. heat conduction). While the classic heat equation $u_t = \Delta u$ accurately describes the heat transfer by conduction in many materials, this model has two important drawbacks. First, the classic heat equation does not take into account the memory effects which may be present in some materials, especially at low temperatures. Most importantly, the heat equation predicts an unrealistic result: that a thermal disturbance at one point of the body is instantly felt everywhere in the body, although not uniformly. In other words in Fourier heat conductors, finite thermal discontinuities must propagate with infinite speed. These observations lead one to believe that in materials with memory, the Fourier's law of heat conduction may be an approximation (perhaps for sufficiently steady temperature fields) to a more general nonlinear constitutive assumption relating the heat flux to the material's thermal history.

Gurtin and Pipkin have proposed a memory theory of heat conduction which is independent of the present value of the temperature gradient, generalizes constitutive relations deduced from kinetic theory by Maxwell and Cattaneo, and has finite heat propagation speeds. The linearized version of Gurkin-Pipkin's model reads as

$$\dot{u}(t,x) + \int_0^\infty \beta(s)\dot{u}(t-s,x)ds = C\Delta u(t,x)$$

which is a particular example of a neutral type differential equation. Similar memory effects emerge in Hodgkin-Huxley model, Dawson-Fleming model of population genetics and other models.

Let $Q$ be linear bounded covariance operator such that $tr(Q) < \infty$.

Introduce

$$W(t) := \sum_{i=1}^{\infty} \sqrt{\lambda_i}\beta_i(t)l_i(x), \ t \geq 0$$

which is a $Q-$Wiener process on $t \geq 0$. Here $\beta_i(t)$ are standard, one dimensional, independent Wiener processes, $\{e_k, k \geq \ 1\}$ is an orthonormal system in K, and a sequence of nonegative real numbers $\lambda_k$ satisfying

$$Qe_k = \lambda_k e_k, \quad k = 1, 2, ...$$

and

$$\sum_{k=1}^{\infty} \lambda_k < \infty.$$

Also let $\{F_t,\ t \geq 0\}$ be a normal filtration satisfying
1)$W(t)$ is $F_t-$ measurable;
2) $W(t+a) - W(t)$ is independent of $F_t$ $\forall a \geq 0, t \geq 0.$
Let $A$ be an infinitesimal generator of an analytic semigroup $\{S(t),\ t \geq 0\}$ in H.

**Condition (H1).**
If $\sigma(-A)$ is the spectrum of $(-A)$, we have $Re\ \sigma(-A) > \delta > 0$, the fractional power $(-A)^\alpha$, which is a closed linear operator with domain $D(-A)^\alpha$.
Denote $H_\alpha$ to be the Banach space $D(-A)^\alpha$ endowed with the norm

$$||u||_\alpha := ||(-A)^\alpha||,$$

which is equivalent to the graph norm of $(-A)^\alpha$. This way $H_0 = 0$. It follows from [. Henry], Sec. 1.4, that if $A^{-1}$ is compact, then $S(t)$ is compact for $t > 0$. Next, it follows from [Pazy], Th. 3.2, p.48 that under the condition $(H1)$ the semigroup $S(t)$ is continuous with respect to uniform operator topology for $t > 0$.

Thus, using [25], Th. 3.3, p.48 we may conclude that the operator A has a compact resolvent. Consequently, using [Henry], Th. 1.4.8, we have the following result:

**Lemma (0.1)**

*Under the condition (H1) the embedding* $H_\alpha \subset H_\beta$ *is compact if* $0 \leq \beta < \alpha \leq 1$.

**Lemma (0.2 Henry Th. 1.4.3)**

*Under the condition (H1), for every* $\alpha \geq 0$ *there exists* $C_\alpha > 0$ *such that*

$$||(-A)^\alpha S(t)|| \leq C_\alpha t^{-\alpha} e^{-\delta t}, \ t > 0. \tag{2}$$

## Lemma (0.3 Da Prato and Zabczyk)

*Let* $p > 2,\ T > 0$ *and let* $\Phi$ *be an* $L_2^0$ *valued, predictable process such that* $E\int\limits_0^T \left|\left|\Phi(t)\right|\right|_{L_2^0}^p dt < \infty$ *Then there is a constant* $M_t > 0$ *such that*

$$E \sup_{t\in[0,T]} \left|\left| \int\limits_0^t S(t-s)\Phi(S)dW(S) \right|\right|^p \leq M_T E \int\limits_0^T \left|\left|\Phi(t)\right|\right|_{L_2^0}^p dt.$$

*The following are the main assumptions on the coefficients of the equation (1):*

**Condition (H2).**

The mappings $f : C_h \to H$ and $\sigma : C_h \to L_2^0$ are continuous and satisfy

[i] There exists a positive constant $K > 0$ such that

$$||f(\phi)|| + ||\sigma(\phi)||_{L_2^0} \leq K\Big(1 + ||\phi||_c\Big) \quad \text{for all} \quad \phi \in C_h.$$

[ii] There is a function $N : [0, +\infty) \to [0, +\infty)$ such that

(a) the function N is continuous, nondecreasing, concaved, and $N(0) = 0$.

(b) For $p > 2$ and for all $\phi_1, \phi_2 \in C_h$ we have

$$||f(\phi_1) - f(\phi_2)||^p + ||\sigma(\phi_1) - \sigma(\phi_2)||^p_{L_2^0} \leq N\Big(||\phi_1 - \phi_2||^p_{C_h}\Big).$$

(c) The only nonnegative function z(t) satisfying

$$z(t) \leq D \int_0^t N(z(s))ds,$$

for all $t \in [0, T]$ is $z \equiv 0$.

**Remark 0.4** The condition (c) holds for any $N$ satisfying (a) and

$$\int_0^1 \frac{dt}{N(t)} = \infty.$$

**Condition (H3)**
There exist positive constants $0 < \alpha \leq 1$ and $0 < M_g < 1$ such that for all $\phi_1, \phi_2 \in C_h$ the function $g : C_h \to H_\alpha$ satisfies

$$||g(\phi_1) - g(\phi_2)||_{H_\alpha} \leq M_g||\phi_1 - \phi_2||_{Ch}.$$

**Condition (H4)**
The initial condition $\varphi : [-h, 0] \times \Omega \to H$ is an $F_0 -$ measurable random variable, independent of $W$, which has continuous trajectories.

## Definition (0.5)

A continuous $F_t$-adapted stochastic process $u : [-h, T] \times \Omega \to H$ is a mild solution for (1) for $t \in [0, T]$ if it satisfies the integral equation

$$
\begin{aligned}
u(t) = S(t)\Big(\varphi(0) + g(\varphi)\Big) - g(u_t)\cancel{\Big)} - \int_0^t AS(t-\overset{s}{\cancel{S}})g(u_{\overset{s}{\cancel{S}}})\,d\overset{s}{\cancel{S}} + \\
+ \int_0^t S(t-\overset{s}{\cancel{S}})f(u_S)\,d\overset{s}{\cancel{S}} + \int_0^t S(t-\overset{s}{\cancel{S}})\sigma(u_{\overset{s}{\cancel{S}}})\,dW(\overset{s}{\cancel{S}}).
\end{aligned}
\tag{3}
$$

and $u(t) = \varphi(t)$ a.s. for $t \in [-h, 0]$.

## Theorem ( 0.6 Existence and uniqueness)

*Suppose the conditions (H1)-(H4) hold. Then for all $T > 0$ the equation (1) has a unique mild solution $u$ on $[0, T]$. Furthermore, this solution satisfies*

$$\left|\left|u_t - u_0\right|\right|_{C_h} = \sup_{\Theta \in [-h,0]} \left|\left|u(t+\Theta) - \varphi(\Theta)\right|\right| \xrightarrow{p} 0,\ t \to 0. \tag{4}$$

In contrast with non-neutraltype equations (with $g \equiv 0$), the primary difficulty of the equation (1) comes from the term $AS(t-s)g(u_s)$. Generally speaking, if $g(u_s) \in H$, this term has a non-integrable singularity $at,\ t = s$. We overcome this difficulty by introducing the fractional powers of the operator $(-A)$. In particular, we show that if $g$ is regular enough for g(u$_s$) $\in D(-A)^\alpha for\ \alpha > 0$, then this singularity becomes integrable.

**Remark 0.7**

In order to prove Theorem 0.6 we will use the methods outlined in [A. Samoilenko, N. Mahmudov and O.Stanzhytskyi 2008], which established the corresponding result for the neutral type stochastic equations in a simpler case. Besides, our Theorem has a different condition on the Lipschitz constant. Namely, [A. Samoilenko, N. Mahmudov and O.Stanzhytskyi 2008] requires

$$4^{p-1}\Big|\Big|(-A)^{-\alpha}\Big|\Big|M_g < 1, \tag{5}$$

while we require a much more easy to check condition $M_g < 1$. Similar conditions, involving fractional powers of $(-A^{-\alpha})$, appear in [Govindan, 2011, Jiang and Shen,2017, Liu,2018] among others. In addition, we relax the regularity condition for the map $g(\phi)$. In particular, we require $g \in H_\alpha$ for $\alpha \in (0, 1)$, while in [A. Samoilenko, N. Mahmudov and O.Stanzhytskyi 2008] the authors require $\alpha \in \left(\frac{1}{p}, 1\right)$ for $p > 2$.

## Theorem ( 0.8 Continuous dependence on the initial data)

*Suppose the conditions of Theorem 0.6 hold.*
*Let* $u(t,\varphi)$ *and* $u(t,\Psi)$ *be two solutions of (1) with the initial conditions* $\phi$ *and* $\Psi$ *satisfying the condition (H4).*
*Then*

$$E\sup_{t\in[0,T]}\left[||u(t,\varphi)-u(t,\Psi)||^p+||u_t(\varphi)-u_t(\Psi)||^p_{C_h}\right]\to 0,$$

*as* $E||\varphi-\Psi||^p_{C_h}\to 0.$

## Theorem (0.10 Markov property)

*Under the assumptions of Theorem 0.6, the process $u_t(s,\varphi)$ is the Markov.*

**Proposition 0.11**

For any $t \geq s \geq 0$ and any $A \in C_h$ we have $P(s,\phi,t,A) = P(0,\phi,t-s,A)$.
For $g \in B_b(C_h)$, for all $\varphi \in C_h$ and $t \geq s \geq 0$ define

$$P_{s,t}(\varphi) = Eg\Big(u_t(s,\varphi)\Big).$$

It is easy to see that in our case we have $P_{s,t}(\phi) = P_{0,t-s}(\varphi)$ and denote $P_t(\varphi) = P_{0,t}(\varphi)$. The following proposition follows from Theorem 0.8.

**Proposition 0.12**

Under the assumptions (H1)-(H4) the transition semigroup $P_t,\ t \geq 0$ is stochastically continuous and and satisfies the Feller property $P_t : C_b(C_h) \to C_b(C_h)$ and

$$\lim_{t \to 0} P_t \varphi = \varphi.$$

## Theorem (0.13)

*Assume the assumptions of Theorem 0.6 (Existence and uniqueness) hold. Suppose the equation (1) has a solution $u(t)$ which is bounded in probability for $t \geq 0$ in $C_h$, namely*

$$\sup_{t \geq 0} P\Big\{||u_t||_{C_h} > R\Big\} \to 0, \ \ R \to \infty. \tag{6}$$

*Then there exists an invariant measure $\mu$ in $C_h$, i.e.*

$$\int_{C_h} P_t \varphi(x) d\mu(x) = \int_{C_h} \varphi(x) d\mu(x)$$

*for any $t \geq 0$, and $\varphi \in C_b(C_h)$.*

## Applications

Let $D$ be a bounded domain in $R^d$ with $\partial D$ satisfying the Lyapunov condition, $H = L^2(D)$, and

$$Au = \sum_{i,j=1}^{d} \Big(a_{ij}(x) u_{x_i}\Big) x_j = \mathrm{div}\,\Big(a(x)\nabla u\Big).$$

Here $a_{i,j}$ are Holder continuous coefficients with the Holder exponent $\beta \in (0,1)$, symmetric, bounded, and satisfies the elliptic condition

$$\sum_{i,j=1}^{d} a_{i,j}\eta_i\eta_j \geq C_0 \Big|\Big|\eta\Big|\Big|^2, \ \eta \in R^d$$

for some $C_0 > 0$.

Let $e_n(x)$ be orthonormal basis in $H$, such that $e_n \in L^\infty(D)$, and
$\sup\limits_u ||e_n||_{L^\infty(D)} < \infty$
Introduce the covariance operator $Q \in L(H)$ such that $Q$ is non-negative, $Tr(Q) < \infty$ and

$$Q_{e_n} = \lambda_n e_n.$$

This enables us to define

$$W(t) := \sum_{i=1}^{\infty} \sqrt{\lambda_i} \beta_i(t) e_i(x), \ t \geq 0$$

which is a $Q-$ Wiener process for $t \geq 0$ with values in $L_2(D)$.

Denote $U := Q^{\frac{1}{2}}(L^2(D))$.

It follows from [ R. Mantey and T. Zausinger, 1999], Lemma 2.2, that
$U \in L^\infty(D)$.

Following [R. Mantey and T. Zausinger, 1999], introduce the multiplication operator
$\Psi : U \to H$ as follows.

For fixed $\phi \in L^2(D)$, let $\Psi(\phi) = \phi \cdot \psi$ for $\psi \in U$. Since $\phi \in L^2(D)$ and $\psi \in L^\infty(D)$, the operator $\Psi$ is well defined, and hence $\Psi \circ Q^{\frac{1}{2}} : L^2 D \to L^2 D$ defines a Hilbert-Schmidt operator, with

$$||\Psi \circ Q^{\frac{1}{2}}||^2_{L_2^0} \leq Tr(Q) \sup_n ||l_n||^2_\infty ||\varphi||^2_{L^2(D)}.$$

Consider the following delay equation:

$$
\begin{aligned}
d\Big[u(t,x)+\int_D b\big(x,u(t-h,y),y\big)\,dy\Big] &= \Big[\operatorname{div}\big(a(x)\nabla u(t,x)\big)+f\big(u(t-h,x)\big)\Big]\,dt+ \\
&\quad +\sigma\big(u(t-h,x)\big)\,dW(t), \qquad (7)\\
u(t,x)=0, &\quad x\in\partial D,\ t\geq 0, \\
&\text{for } t\in[-h,0].
\end{aligned}
$$

Here $b(x,z,y): R^d\times R^1\times R^d\to R,\quad f:R\to R$ and $\sigma:R\to R$..
Introduce the mapping

$$
g(\varphi)(x):=\int_D b\Big(x,\varphi(-h,y)\Big)dy,
$$

as a map from $C_h \ \to L^2(D)$.

Then the problem (7) can be viewed in the abstract form (1), with

$$
D(A)=H^2(D)\cap H_0^1(D)
$$

Assume the following conditions hold:
**(C1)** the fucntions $f$ and $\sigma$ satsify the Lipschitz condition with Lipschitz constant $L$.
**(C2)** the function b is continuous with respect to all of its variables, and there exist constants $A > 0$ and $M > 0$ such that

$$|b(x, 0, y)| + |\nabla_x b(x, y, z)| \leq A,$$

and

$$|b(x, z_1, y) - b(x, z_2, y)| + |\nabla_x b(t, z_1, y) - \nabla_x b(x, z_2, y)| \leq M_g |z_1 - z_2|,$$

for all $x, y \in D, \quad z_1, z_2 \in R$, and

$$2M_g^2 meas^2(D) < 1.$$

It is then easy to verify that all the conditions of the existence theorem are satisfied. If we assume that

$$|f| + |\sigma| + |b| + |\nabla_x b| \leq C,$$

Then the condition of **Theorem 0.13** hold, and there exists invariant measure.

## Weak solutions of neutral stochastic functional differential equations in infinite dimensional spaces

Let $K, H, V$ be real separable Hilbert spaces, $V \subset H \subset V'$ be the Gelfand triple, where the inclusion $V \subset H$ is dense and compact, and $V'$ is dual to $V$. The norm in $V$ we denote by $||\cdot||_V$, the inner product in $H$ we denote by $\Big(\cdot,\cdot\Big)$, and $\Big\langle \cdot,\cdot \Big\rangle$ denotes the pairing between $V$ and $V'$.

Assume that $A : V \to V'$ is a self-adjoint and coercive linear operator and $a\Big(u,v\Big) = \Big\langle Au, v\Big\rangle, u, v \in V$ is the corresponding bilinear form.
We denote $\Omega_T := \Omega \times [0,T]$.

## Definition

For a given $F_0$ measurable random variable $\phi \in C_h$ an $F_t$ adapted random process $u(t) \in V$ is a weak solution of $(1), (2)$ on $[0, T]$ if:

$1. u(t) = \varphi(t), t \in [-h, 0],$

$2. u \in L_2\Big(\Omega_T; V\Big) \cap L^2\Big(\Omega; C([0, T]; H)\Big),$

3. for any $v \in V$ we have

$$\Big(u(t) - g(u_t), v\Big) = \Big(\phi(0) - g(\varphi), v\Big) + \int\limits_0^t \Big\langle Au(s), v\Big\rangle ds + \int\limits_0^t \Big(f(u_s), v\Big) ds + \int\limits_0^t \Big(\sigma(u_s), \overset{v}{\cancel{\varphi}}\Big) dW(s)$$

for each $t \in [0, T]$ a.s.

## Conditions on $A$:

(A1) $A : V \to V'$ is a linear bounded self-adjoint operator.

(A2) $\exists \alpha > 0$ such that for any $u, v \in V$ we have

$$\left|\left\langle Au, v \right\rangle\right| \le \alpha \left\|u\right\|_V \cdot \left\|v\right\|_V.$$

(A3) $A$ satisfies the coercivity condition: $\exists \beta > 0$ such that

$$\left\langle Av, v \right\rangle \le -\beta \left\|v\right\|_V^2, \ \forall v \in V.$$

Let $L_2^V = L^2\Big([-h;0);V\Big)$ with the norm

$$\left\|u\right\|_{L_2^V}^2 := \int_{-h}^{0} \left\|u(t)\right\|_V^2 dt.$$

**Conditions on $g$:**

(G1) $g$ is the mapping from $C \cap L_2^V$ to $V$;

(G2) (Growth condition):

a) $\exists K_1 > 0$, such that for all $\varphi \in C \cap L_2^V$ we have

$$||g(\varphi)||_V \leq K_1\Big(1 + ||\varphi||_{L_2^V} + ||\varphi||_C\Big)$$

b) $\exists K_2 > 0,\ \lambda \in (0,1)$ such that for all $\varphi \in C \cup L_2^V$ we have

$$||g(\varphi)||^2 \leq K_2(1 + \int\limits_{-h}^{0} ||\varphi(\Theta)||^2 d\Theta) + \lambda||\varphi(0)||^2.$$

(G3) $\exists L \in \left(0, \frac{1}{4(h+1)}\right)$, s.t. $\forall \varphi, \psi \in C \cap L_2^V$

$$||g(\varphi) - g(\psi)||^2 \leq L\Big( \int\limits_{-h}^{0} ||\varphi(\Theta) - \psi(\Theta)||^2 d\Theta + ||\varphi(0) - \psi(0)||^2 \Big).$$

(G4) $\exists L_2 \in \left(0, \frac{\beta}{\alpha}\right)$, s.t. $\forall \varphi, \in C \cap L_2^V$

$$||g(\varphi) - g(\psi)||_V \leq L_2\Big( \int\limits_{-h}^{0} ||\varphi(\Theta) - \psi(\Theta)||_V d\Theta + ||\varphi(0) - \psi(0)||_V \Big) + (\rho(\varphi) + \rho(\psi))||\varphi - \psi||_C$$

where $\rho : C \cap L_2^V \to [0, \infty)$, and $\exists K_3 > 0,\ \exists \gamma \geq 1$ s.t.

$$\rho(\varphi) \leq K_3\Big( ||\varphi||_C^\gamma + \Big( \int\limits_{-h}^{0} ||\varphi(\Theta)||_V d\Theta \Big)^\gamma \Big).$$

**Conditions on $f$ and $\sigma$:**

$(N1)$ $f$ is a mapping from $C \cap L_2^V \to H$,

$\sigma$ is a mapping from $C \cap L_2^V \to L_2^0$.

$(N2)$ (Growth condition). $\exists K_4, K_5 > 0,\ \gamma \geq 1$ s.t.

$$||f(\varphi)|| \leq K_4\Big(1 + \Big(\int_{-h}^{0} ||\varphi(t)||_V dt\Big)^{\gamma} + ||\varphi||_C^{\gamma} + ||\varphi(0)||_V^{\gamma}\Big)$$

$$||\sigma(\varphi)||_{L_2^0}^2 \leq K_5\Big(1 + ||\varphi||_C^2\Big) \quad \text{for all} \quad \varphi \in C \cap L_2^V.$$

$(N3)$ $\exists \nu > 0$ s.t. $\forall \varphi \in C \cap L_2^V$

$$\Big(f(\varphi), \varphi(0)\Big) \leq \nu\Big(||\varphi||_C^2 + 1\Big)$$

$(N4)$ (Monotonicity condition).
$\exists \delta > 0$ s.t. $\forall \varphi, \psi \in C \cap L_2^V$

$$||\sigma(\varphi) - \sigma(\psi)||^2_{L_2^V} \leq \delta ||\varphi - \psi||^2_C$$

$$\Big(f(\varphi) - f(\psi), \varphi(0) - \psi(0)\Big) \leq \Big(\rho(\varphi) + \rho(\psi)\Big)||\varphi - \psi||^2_C$$

$$||f(\varphi) - f(\psi)|| \leq \Big(\rho(\varphi) + \rho(\psi)\Big)\Big(||\varphi(0) - \psi(0)||_V + \int\limits_{-h}^{0} ||\varphi(\Theta) - \psi(\Theta)||_V d\Theta\Big),$$

where $\rho$ is given in (G4)

$N5$) $\exists K_6 > 0$ s.t. $\Big(f(\varphi), g(\varphi)\Big) \leq K_6\Big(1 + ||\varphi||^2_C\Big)$.

## Theorem (5. Existence and uniqueness weak solution)

*Assuming some condition hold, for any $F_0$ measurable initial condition $\phi \in C \cap L^2_V$ which satisfies the condition*

$$E\Big(||\phi||_C^{2\gamma} + \Big(\int_{-u}^{0} ||\phi(\Theta)||_U^2 d\Theta\Big)^{\gamma}\Big) < \infty,$$

*the initial value problem (1)-(2) has a unique weak solution on $[0, T]$ such that*

$$E\Big(\sup_{t\in[0,T]} ||u(t)||^{2\gamma} + \Big(\int_{-u}^{0} ||\phi(\Theta)||_U^2 d\Theta\Big)^{\gamma}\Big) < \infty,$$

Moreover, the following energy equality holds almost surely:

$$||u(t) - g(u_t)||^2 = ||\phi(0) - g(\phi)||^2 + 2\int_0^l \Big\langle Au(s), u(s) - g(u_s)\Big\rangle ds +$$

$$+2\int_0^t \Big(f(u_s), u(s) - g(u_s)\Big) ds + \int_0^t ||\sigma(u_s)||^2_{L_2^0} ds + 2\int_0^t \Big(\sigma(u_s) dW(s), u(s) - g(u_s)\Big).$$

## Application.

We consider the nonlocal stochastic reaction-diffusion functional-differential neutral type equation

$$
\begin{aligned}
&d\left(u(t,x) + g\left(x, \int_D \Big(\int_{-h}^{0} u(t+\Theta, y)\, d\Theta\Big) dy, \int_D u(t,y)\, dy\right)\right) = \\
&\quad = \left(Au(t,x) + f\left(x, \int_D \Big(\int_{-h}^{0} u(t+\Theta, y)\, d\Theta\Big) dy\right) - \Big(\int_D u(t,y)\, dy\Big)^3\right) dt + \\
&\quad + \sigma\left(x, \int_{-h}^{0} u(t+\Theta, y)\, d\Theta, u(t,x)\right) dW(t), \\
&u(t,x) = \varphi(t,x), \quad t \in [-h, 0], \\
&u(t,x) = 0, \quad x \in \partial D,\ t \geq 0.
\end{aligned}
\tag{8}
$$

Assume the functions $g, f$ and $\sigma$ are continuous, and the following conditions hold:

1. There exists constant $K > 0$ such that

$$\sigma^2(x, u, v) \leq K(1 + u^2 + v^2),$$

for all $x \in D, \quad u, v \in R$, and

$$\Big(\sigma(x, u, v) - \sigma(x, u_1, v_1)\Big)^2 \leq K\Big((u - u_1)^2 + (v - v_1)^2\Big),$$

for all $x \in D, u, v, u_1, v_1 \in R$.

2. The functions $g(x, u, v)$ and $g_x(x, u, v)$ are continuous and globally Lipshitz with respect to the variables u and v. Furthermore, assume that the Lipshitz constant $L > 0$ satisfies

$$\mu(D)L^2 < \frac{1}{4(h + 1)}$$

In addition, assume that for some $K_1 > 0$ and $\lambda \in (0, 1)$ we have

$$g^2(x, u, v) \leq K_1(1 + u^2) + \lambda v^2,$$

for all $x \in D, \quad u, v \in R$.

3. There exists $C > 0$ such that

$$|f(x,u) - f(x,u_1)| \leq C|u - u_1|, \text{ for all } \; x \in D, u, u_1 \in R.$$

4. There exists some $K_2 > 0$ such that

$$g(x,u,v) \cdot \Big(f(x,u) - v^3\Big) \leq K_2(1 + u^2 + v^2),$$

for all $x \in D, u, v \in R$.

## Theorem (3)

*Suppose the conditions 1)-4) hold. Then for every $F_0-$measurable $\varphi$ such that*

$$E\Big(||\varphi||_{C_h}^{2\gamma} + \Big(\int_h^0 ||\varphi(t)||_v^2 dt\Big)^{\gamma}\Big) < \infty \tag{9}$$

*the statements of Theorems 5 are true.*

## Invariant measures and their limiting behaviour for neutral type stochastic delay equations in Hilbert space.

The main goal is to study the limiting behaviour of invariant measures for neutral type stochastic delay equations in Hilbert space of the form

$$\begin{aligned} d\Big(u(t) - g(u(t-h))\Big) &= \Big(Au(t) + f(u(t-h), u(t))\Big)\, dt + \\ &\quad + \sigma(u(t-h), u(t))\, dW(t), \\ u(t) &= \varphi(t), \quad t \in [-h, v], \quad t > 0. \end{aligned} \tag{10}$$

To ensure the existence and uniqueness of solution we have to impose additional conditions on the operator A and mappings $f, \sigma, g$.

1. If $\sigma(-A)$ is a spectrum of $(-A)$, we have that $Re\sigma(-A) > \delta > 0$, and $A$ generates a semigroup of compact operators $S(t)$ in $H$.

2. For all $u, v, u_1, v_1 \in H$ we have

$$||f(u,v) - f(u_1,v_1)|| \leq L\Big(||u - u_1|| + ||v - v_1||\Big),$$

and

$$||\sigma(u,v) - \sigma(u_1,v_1)||_{L_2^0} \leq L\Big(||u - u_1|| + ||v - v_1||\Big)$$

for some $L > 0$.

3. There exists $\alpha \in \left(\frac{1}{2}, 1\right)$ such that for all $u, u_1 \in H$ the function $g$ satisfies

$$||g(u) - g(u_1)||_\alpha \leq M_g||u - u_1||$$

for some $M_g \in \left(0, \frac{1}{2}\right)$.

For any h we study the limiting behaviour of the family of invariant measures $\{\mu_h\}$ as $h \to 0$ in the sense of weak convergence. More precisely, we prove that under some conditions the family $\{\mu_h\}$ is tight, and any limit point of $\{\mu_h\}$ is an invariant measure of the corresponding limiting system $(h = 0)$

$$
\begin{aligned}
d\Big(u^0(t) - g(u^0(t))\Big) &= \Big(Au^0(t) + f(u^0(t), u^0(t))\Big)\, dt + \\
&\quad + \sigma(u^0(t), u^0(t))\, dW(t), \\
u^0(0) &= \varphi(0).
\end{aligned}
\tag{11}
$$

Additionally, we prove that if system (10) has a unique invariant measure, which is stable, then $\mu_{h_n} \to \mu_0$ for an arbitrary sequence $h_n \to 0$. In this case not only the question of existence of invariant measure becomes of significant importance, but the question of uniqueness as well. This question is also studied in the talk. In the sequel, we will always assume that the initial conditions are deterministic. Our main result about convergence of $\mu_h$ as $h \to 0$ will be based on the following auxiliary arguments.

For every $h \in (0,1]$ and $\phi \in C\Big([-h,0],H\Big)$ let $u^h(t,\phi)$ be a solution of (10) and it is a stochastic process in the state space $C\Big([-h,0],H\Big)$. Similarly, for every $u_0 \in H, u^0(t,u_0)$ is a solution of (11) such that $u^0(0,u_0)=u_0$. For a given $h \in (0,1]$ we define operators

$T_h : C\Big([-h,0],H\Big) \to H,$

$T_h(\varphi) := \varphi(0)$ for $\phi \in C\Big([-h,0],H\Big)$, and

$\Gamma_h : C\Big([-1,0],H\Big) \to C\Big([-h,0],H\Big),$

$\Gamma_h(\varphi)(s) := \varphi(s)$ for $\varphi \in C\Big([-1,0],H\Big)$.

Next, we assume

$$\lim_{h\to 0} \sup_{\varphi \in \Gamma_h K} P\Big\{\big\|u^h(t,\varphi)-u^0(t,T_h\varphi)\big\| \geq \eta\Big\} = 0 \tag{12}$$

for all compact subsets $K$ from $C\Big([-1,0],H\Big)$, for all $h>0$, and for all $t \geq 0$.

## Theorem (6. 2, Th. 7.1)

*Assume that (12) holds, and* $h_n \subset (0,1]$. *Let* $\mu h_n$ *be an invariant measure of* $u_{h_n}$ *in* $C\Big([-h_n, 0], H\Big), n \geq 1$.

*Suppose, that* $\{\mu_{h_n}{}^{\infty}\}$

*is tight in the sense, that for every* $\epsilon > 0$ *there exists a compact set* $K_1 \subset C\Big([-1,0], H\Big)$ *such that for all* $n \geq 1$

$$\mu_{h_n}\Big\{\mathit{\Gamma}_{h_n K_1}\Big\} > 1 - \epsilon \tag{13}$$

Then

1. the sequence $\left\{\mu_{h_n} \circ T_{\mu_{h_n}}^{-1}\right\}_1^{\infty}$ is tight;

2. if $h_n \to 0$ and $\mu$ is a probability measure in $H$ such that $\mu_{h_n} \circ T_{\mu_{h_n}}^{-1} \to \mu$ weakly, then $\mu$ is an invariant measure of $u_0$.

Let $M_h$ be the set of invariant measure for (10).

## Theorem (7)

*Suppose conditions 1 - 3 hold. Then*

*1. The union* $\cup_h \in [0,1] \quad M_h$ *is tight.*

*2. If* $h_n \to 0$ *and* $\mu_{h_n} \in M_{h_n}$*, then there exists subsequence* $h_{n_k}$ *and invariant measure* $\mu_0 \in M_0$ *such that* $\mu_{h_{n_k}} \circ T_{\mu_{h_{n_k}}}^{-1} \to \mu_0$ *weakly.*

*Finally, as a direct corollary of Theorem 7 we have the following result.*

## Theorem (8)

*Suppose conditions on Theorem 7 hold and the set of invariant measures* $M_0$ *contains only one element* $\mu_0$, *which means the uniqueness of invariant measure for (11). Then*
$\mu_{h_n} \to \mu_0$ *weakly, as* $h \to 0$.

Firstly, we construct Wiener process on $R$ as follows:

$$W(t) = \sum_{i=1}^{\infty} \sqrt{\lambda_i} l_i \beta_i(t),$$

where $\beta_i(t)$ is a Wiener process in $R$ :

$$\beta_i(t) = \begin{cases} \beta_i^+(t), & t \geq 0 \\ \beta_i^-(t), & t < 0, \end{cases}$$

here $\beta_i^+$, $\beta_i^-$ are independent standard one-dimensiotnal Wiener processes for $t \geq 0$.
Also, let

$$F_t := \sigma\Big\{\beta_i(u) - \beta_i(v)\Big\} : \quad v \leq u \leq t, \quad i \geq 1.$$

## Definition

We say that $H-$ valued process $u(t)$ is a mild solution of (1) on $R$ if:

1. For all $t \in R$, $u(t)$ is $F_t-$measurable.
2. $u(t)$ is continuous a.s. in $t \in R$ :
3. For all $t \in R$, $E||u(t)||^2 < \infty$.
4. For all $-\infty < t_0 < t < \infty$ we have

$$u(t) = S(t-t_0)\Big(u(t_0) - g(u(t_0-h)) - g(u(t-h)) - \int\limits_{t_0}^{t} AS(t-t_0)g(u(s-h))ds +$$

$$+ \int\limits_{t_0}^{t} S(t-s)f(u(s-h), u(s))ds + \int\limits_{t_0}^{t} S(t-s)\sigma(u(s-h), u(s))\Big)dW(s)$$

### Theorem (9)

*Let Lipshitz constant small enough. Then there exist solution $u^*(t)$ of equation (10) such that*

$$\sup_{t \in R} E||u^*(t)||^2 < \infty.$$

**Step 1. Linear case.**

Let $B$ be the class of $H-$valued $F_t-$measurable random processes $\xi(t)$, defined on $R$ such that

$$\sup_{t \in R} E||\xi(t)||^2 < \infty.$$

Let $B_\alpha$ be the class of $H_\alpha-$valued $F_t-$measurable random processes $\xi(t)$ on $R$. s.t.

$$\sup_{t \in R} E||\xi(t)||^2_\alpha < \infty,$$

and $B_{HS}$ be the class $F_t-$measurable $L_2^0-$valued random ptocesses $\xi(t)$ on $R$ s.t.

$$\sup_{t \in R} E||\xi(t)||^2_{L_2^0} < \infty.$$

For $F$ in $B$, $\sum \in B_{HS}$, $G \in B_\alpha$ we consider linear equation

$$d\Big(u(t) - G(t)\Big) = \Big(Au + F(t)\Big)dt + \sum(t)dW(t). \tag{14}$$

## Lemma

*Equation (11) has a unique solution $u^* \in B$.*

**Step 2. Non-linear case.**

Let us introduce the sequence of processes $\{u_n(t)\},\ u_0(t) := 0$ and

$$d\Big(u_{n+1}(t) - g(u_n(t-h))\Big) = \Big(Au_{n+1}(t) + f(u_n(t-h), u_n(t))\Big)\,dt + \\ + \sigma(u_n(t-h), u_n(t))\,dW(t). \tag{15}$$

For any $n$ (12) has unique solution $u_{n+1}(t) \in B$.

Then, passing to the limit, we obtain the proof.

**Application.**

$$
\begin{aligned}
d\Big[u(t,x)+\int_D b\Big(x,u(t-h,y),y\Big)\,du\Big] = \Big[\operatorname{div}\big(a(x)\nabla u(t,x)\big)+f(t-h,x),u(t,x)\Big]\,dt+ \\
+\,\sigma\big(u(t-h,x),u(t,x)\big)\,dW(t)
\end{aligned}
\tag{16}
$$

$u(t,x)=\varphi(t,x),\ t\in[-h,0]$, and $u(t,x)=0,\ x\in\partial D,\ t\geq 0$
Let $D$ be a bounded domain in $R^d$, $H=L^2(D,R)=L^2(D)$
$Au=\operatorname{div}\Big(a(x)\nabla u\Big)=\sum_{i,j=1}^{d}\Big(a_{ij}(x)u_{x_i}\Big)_{x_j}$,
$a_{ij}(x)$ are Hölder continuous, with exponent $\beta\in(0,1)$,
symmetric, bounded, and elliptic condition

$$\sum_{i,j=1}^{d} a_{ij}\eta_i\eta_j \geq C_0||\eta||^2,\ \eta\in R^d,\ C_0>0.$$

$b(x,z,y): R^d\times R\times R^d\to R,\ f(u,v): R^2\to R,$ and $\sigma(u,v): R^2\to R.$

For any $v \in L^2(D)$ introduce the mapping

$$g(v)(x) := \int\limits_D b\Big(x, v(y), y\Big) dy : L^2(D) \to L^2(D).$$

**Condition.**

1. The function $f, \sigma$ satisfy the Lipshitz condition with constant $L$.
2. The function $b$ is continuous with respect to all variables and there exist $A > 0$ such that

$$|b(x, 0, y)| + |\nabla_x b(x, z, y)| + ||D_x^2 b|| \leq A.$$

and $b, \nabla_x b, D_x^2 b$ satisfies the Lipshitz condition on $z$, $L-$ constant.
Let the $L$ is small enough, then for the equation (13) exist invariant measure for all $h \in [0, 1)$ and Theorem 9 hold.

Thank you for your attention.


ANY
QUESTIONS?!