Introduction to input-to-state stability

Sergey Dashkovskiy

Julius-Maximilians-Universität Würzburg

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Outline

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Motivating examples

$$ma = F$$

$$m\ddot{x} = -\mu(\dot{x}) - k(x)$$

Denote $x_1 = x$ and $x_2 = \dot{x}$ then

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -k(x_1) - \mu(x_2) \end{cases}$$

The trivial solution is GAS.

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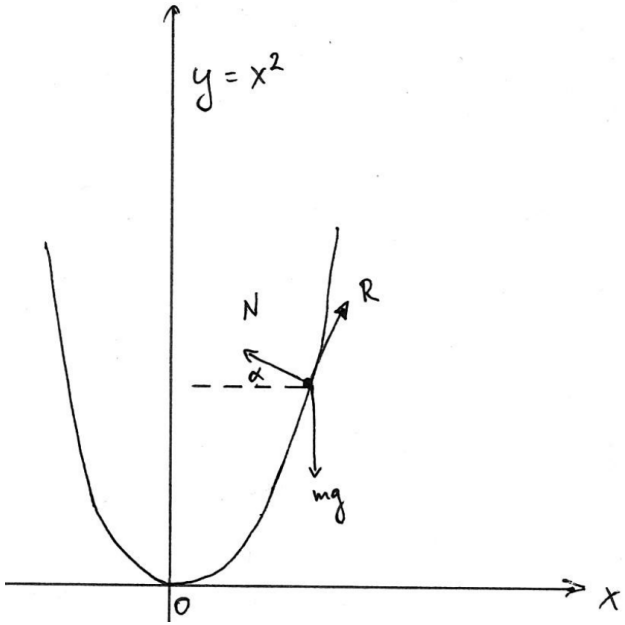

Motion of a mass point along $y = x^2$ with friction

$$(1 + 4x^2)\ddot{x} + 4x\dot{x}^2 + 2gx = 0$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{4x_1x_2^2}{1+4x_1^2} - \frac{2gx_1}{1+4x_1^2} \end{cases}$$

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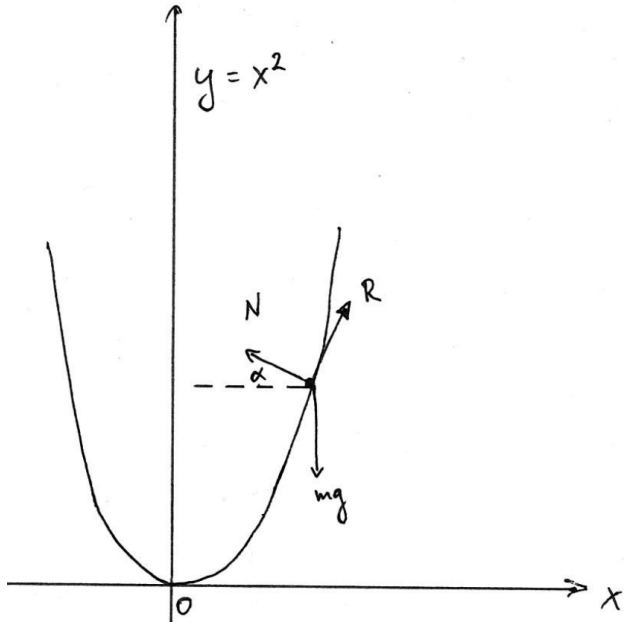

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Motivating questions

Let $\dot{x} = f(x, 0)$ be GAS. What about $\dot{x} = f(x, u)$ for $u \in L_\infty$?

- $\exists\, x(t)$ for all t?
- is $x(t)$ bounded?
- u is "small" $\implies$ $x(t)$ is "small"?
- $\lim_{t\to\infty} u(t) = 0 \implies \lim_{t\to\infty} x(t) = 0$?

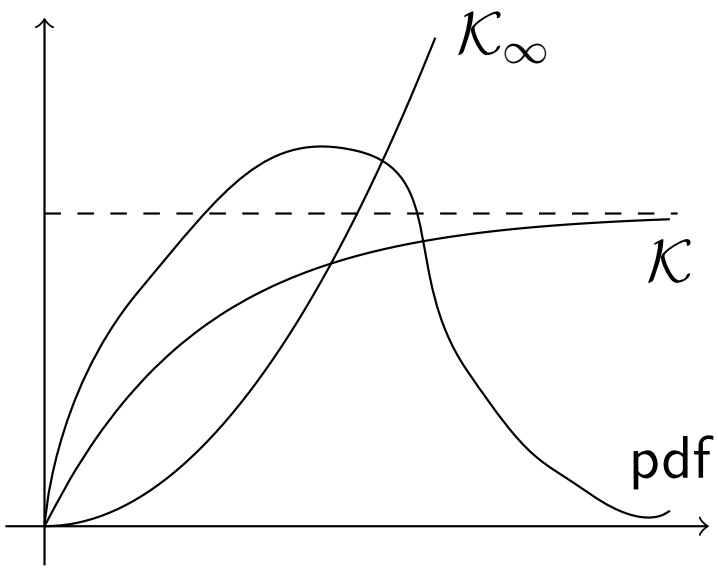
For linear systems: YES. In general: NOT.

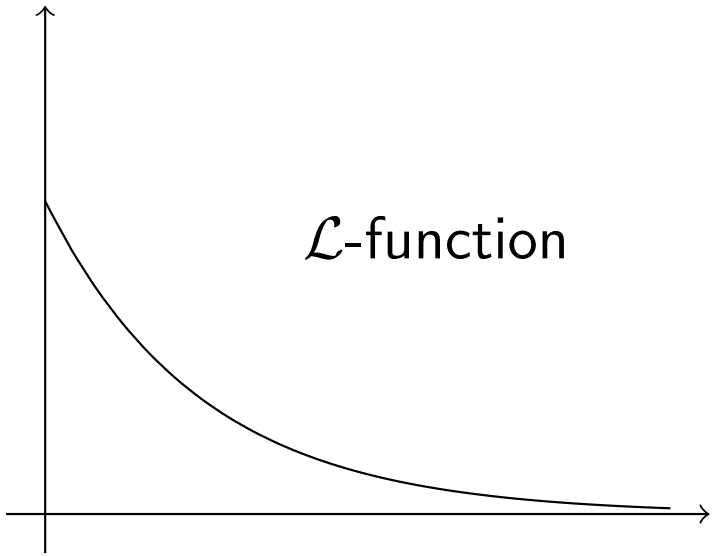
For input-to-state stable systems (ISS): YES.

Weaker notions exist.

Comparison functions

Denote $\mathbb{R}_+ := [0, \infty)$ and consider classes of continuous functions:





- $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is positive definite if $\alpha(r) = 0 \quad \Leftrightarrow \quad r = 0$
- $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{K}$ if it is pos.def. and strict. increas.
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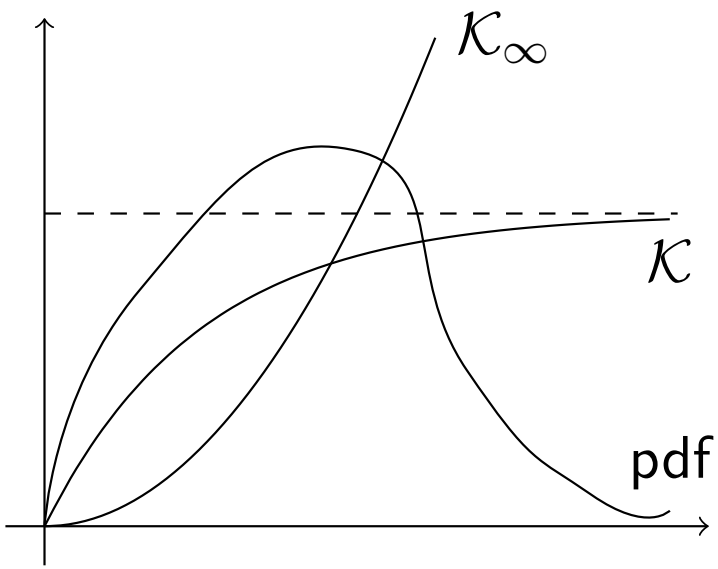

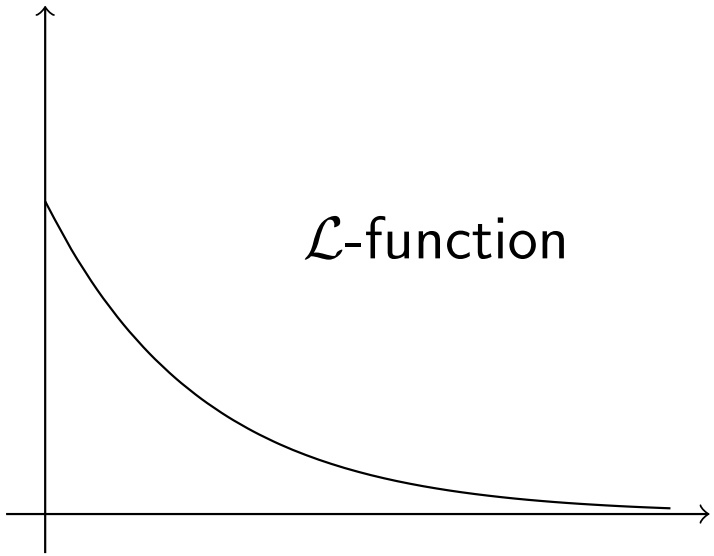

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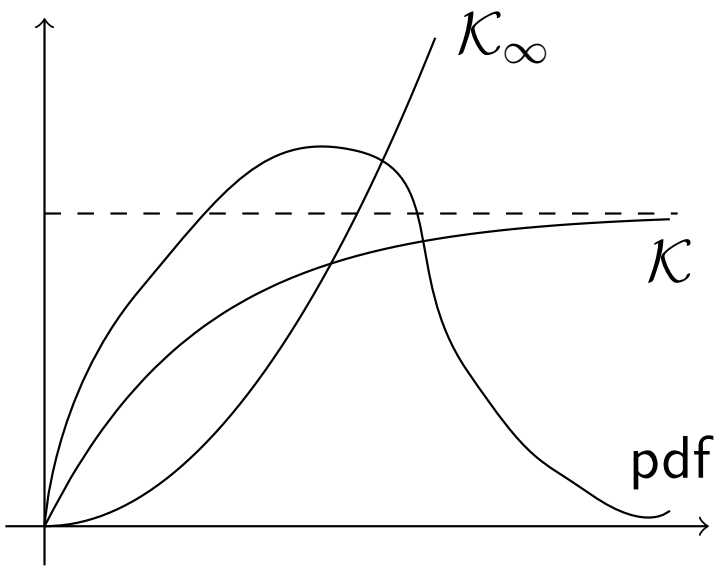

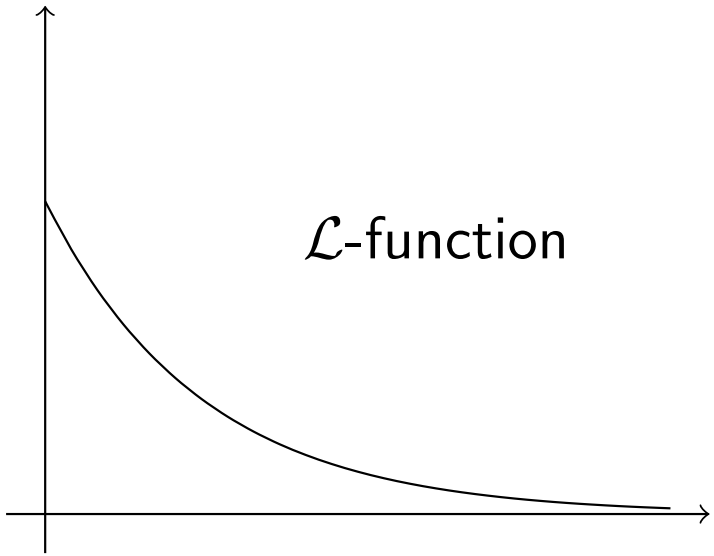

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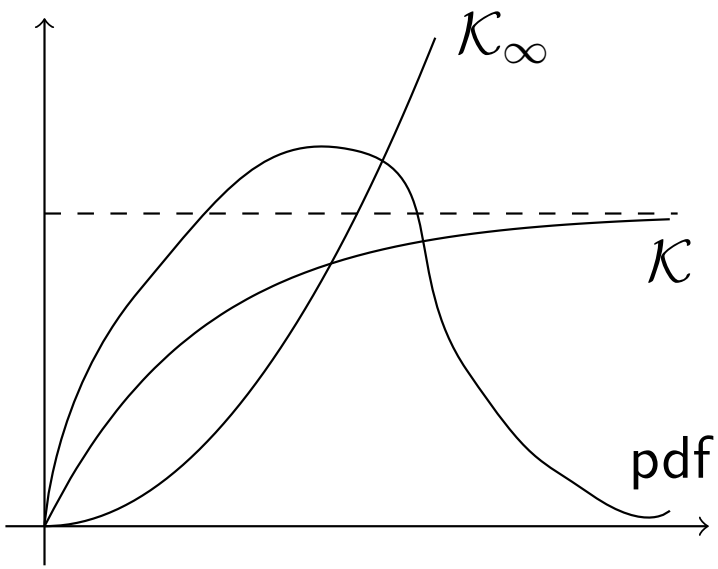

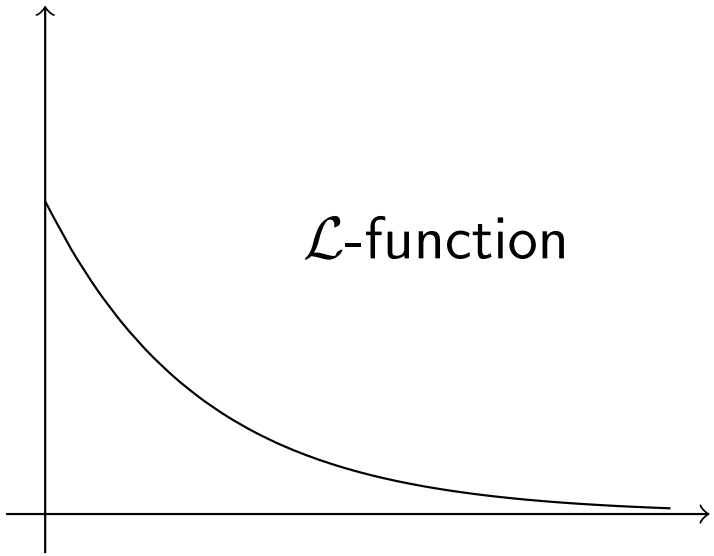

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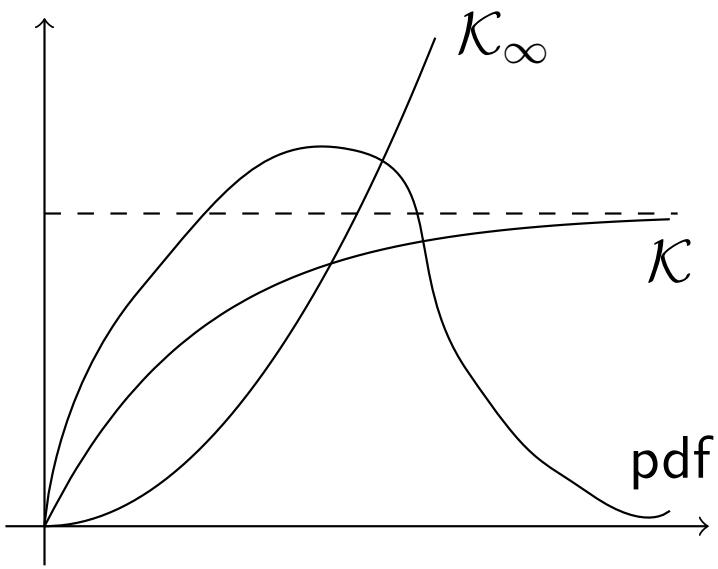

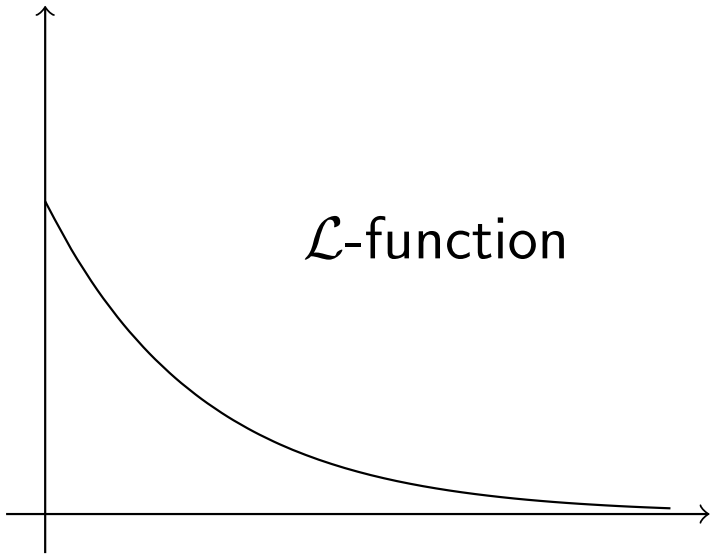

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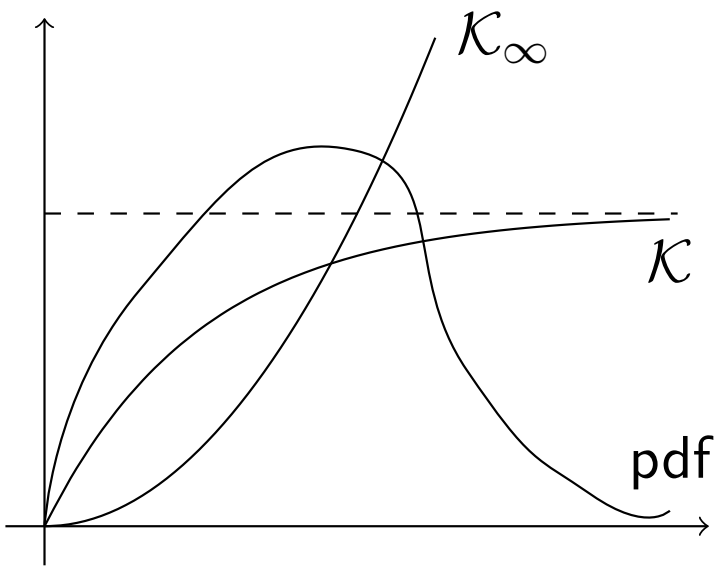

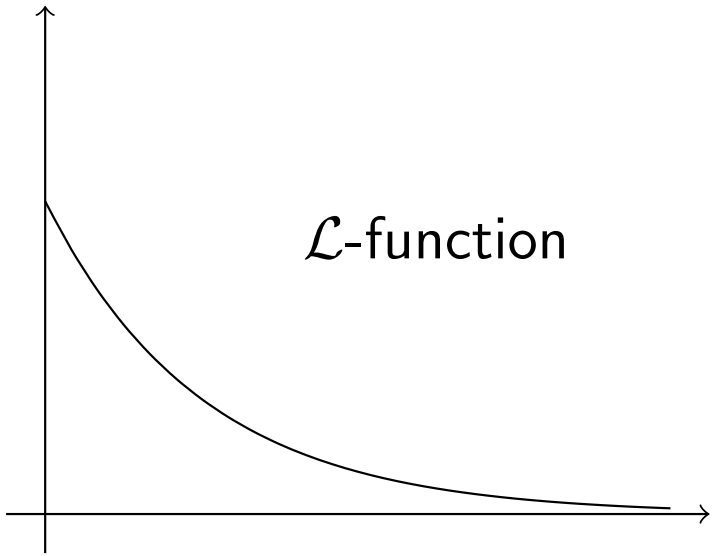

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η is called locally asymptotically stable for $\dot{x} = f(t, x)$, if

- η is stable that is $\forall \varepsilon \; \exists \delta$:
  $||\xi(t_0) - \eta(t_0)|| < \delta \implies ||\xi(t) - \eta(t)|| < \varepsilon, \quad t \geq t_0$
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If $\Delta = \infty$, then η is called globally asymptotically stable (GAS).

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Input-to-state stability (ISS)

Theorem 2

$\dot{x} = f(x)$ is GAS $\quad\Leftrightarrow\quad$ *$\exists\, \beta \in \mathcal{KL}$ such that for any $x(0) \in \mathbb{R}^n$ it holds*

$$|x(t)| \leq \beta(|x(0)|, t), \quad t \geq t_0.$$

Definition 3 (Sontag 1989)

$$\dot{x}(t) = f(x(t), u(t)) \quad \text{is} \quad \text{ISS} \quad :\Leftrightarrow$$

$$\exists\, \beta \in \mathcal{KL} \quad \exists\, \gamma \in \mathcal{K}_\infty$$

$$\forall\, x(0) \in \mathbb{R}^n \qquad \forall\, u \in L_\infty$$

$$|x(t)| \;\leq\; \beta(|x(0)|, t) + \gamma(||u||_\infty), \quad t \geq 0$$

Weeker notions: intergral ISS, local ISS, ...

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- in particular $f(0, 0) = 0$
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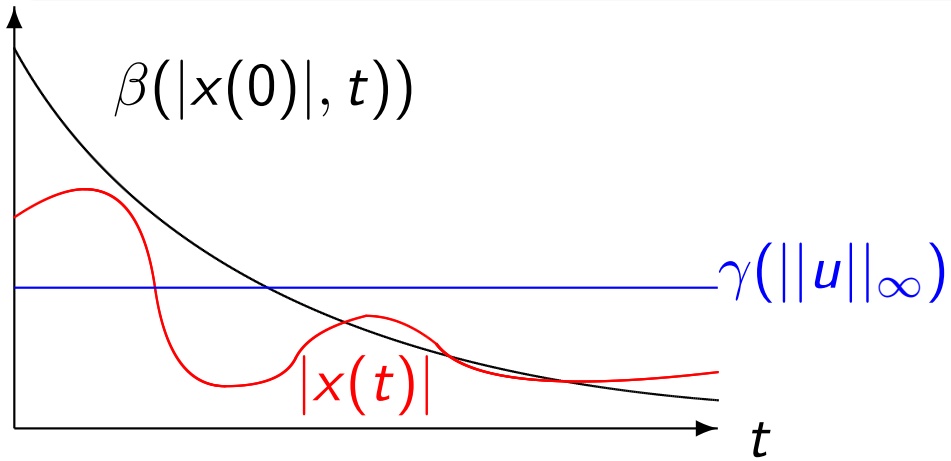

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ISS-Lyapunov function

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$V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is ISS-Lyapunov function for $\dot{x} = f(x, u)$

$:\Leftrightarrow \qquad \exists\, \psi_1, \psi_2, \chi, \alpha \in \mathcal{K}_\infty$

- $\psi_1(|x|) \leq V(x) \leq \psi_2(|x|), \quad x \in \mathbb{R}^n,$
- $V(x) \geq \chi(|u|) \quad \Rightarrow \quad \nabla V(x)\, f(x, u) \leq -\alpha(V(x)).$

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Other characterizations of ISS

Definition 7

- $\dot{x} = f(x,u)$ is called 0-GAS if $u = 0$ implies GAS
- $\dot{x} = f(x,u)$ has the asymptotic gain (AG) property if $\exists \gamma \in \mathcal{K}_\infty : \ \overline{\lim}_{t\to\infty} |x(t)| \leq \gamma(||u||_\infty)$
- $\dot{x} = f(x,u)$ is called globally stable (GS) if $\exists \gamma, \sigma \in \mathcal{K}_\infty : \ |x(t)| \ \leq \gamma(|x(0)|) + \sigma(||u||_\infty)$

$$|x(t)| \ \leq \ \beta(|x(0)|,t) + \gamma(||u||_\infty) \quad \Rightarrow \quad \begin{cases} |x(t)| \ \leq \ \beta(|x(0)|,t) \text{ if } u = 0 \\ \overline{\lim}_{t\to\infty} |x(t)| \leq \gamma(||u||_\infty) \end{cases}$$

The converse is also true: ISS $\Leftrightarrow$ AG and 0-GAS

Moreover, ISS $\Leftrightarrow$ AG and GS

See Sontag and Wang TAC 1996 for more characterizations.

See also Grüne (2002) for ISS $\Leftrightarrow$ ISDS

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The converse is also true: ISS $\Leftrightarrow$ AG and 0-GAS

Moreover, ISS $\Leftrightarrow$ AG and GS

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Equivalent characterization of the ISS LF

Theorem 8

$\dot{x} = f(x,u)$ *is ISS* $\quad\Leftrightarrow\quad$ $\exists \alpha_i \in \mathcal{K}_\infty,\ i = 1,2,3,4$ *such that*

$$\alpha_1(|x|) \le V(x) \le \alpha_1(|x|), \quad x \in \mathbb{R}^n$$

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If α is only pos. def. then initegral ISS property follows so that

$$\rho(|x(t)|) \le \beta(|x_0|, t) + \int_0^t \gamma(|u(s)|)\, ds,$$

for some $\rho, \gamma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$
A more weaker property is local ISS:

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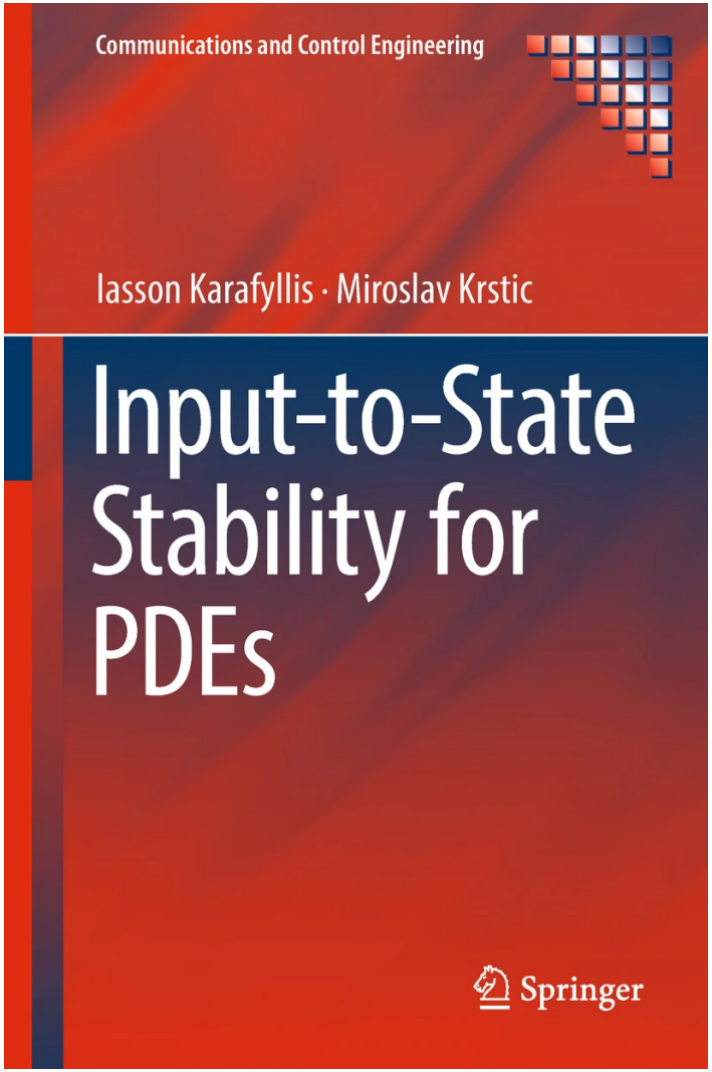
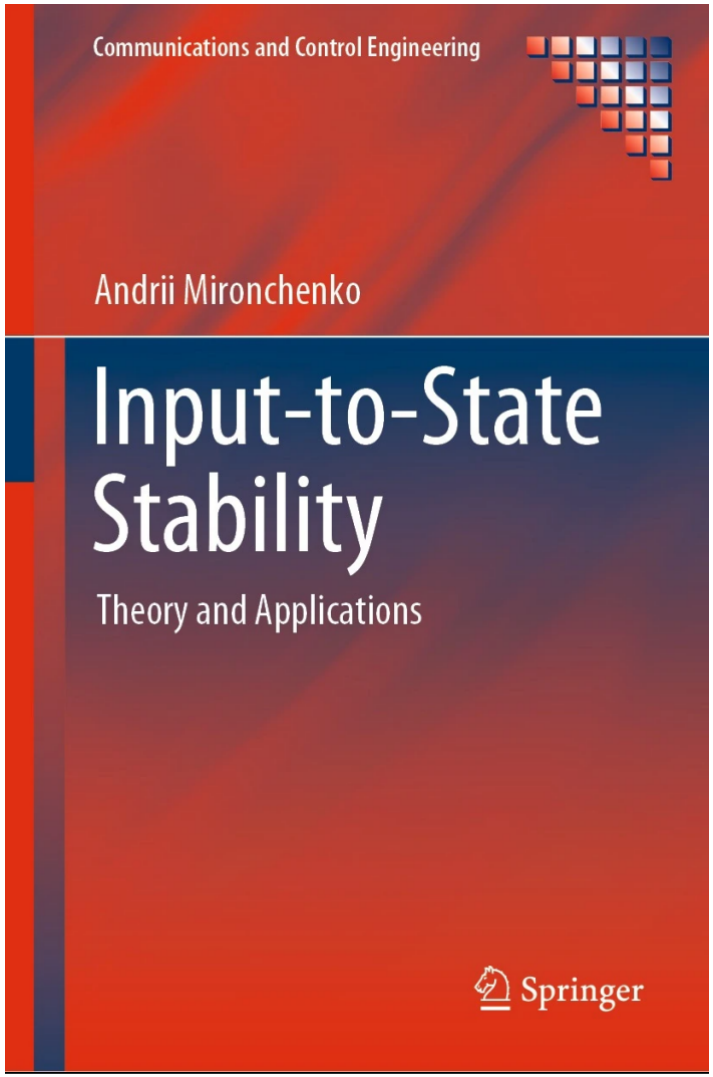

Communications and Control Engineering
Iasson Karafyllis · Miroslav Krstic
Input-to-State Stability for PDEs
Springer


Communications and Control Engineering
Andrii Mironchenko
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Theory and Applications
Springer

Outline

Interconnections

Let the origin $x = 0$ be stable for the system

$$\dot{x} = f_1(x), \quad f_1 : \mathbb{R}^n \to \mathbb{R}^n,$$

Let the origin $y = 0$ be stable for the system

$$\dot{y} = f_2(y), \quad f_2 : \mathbb{R}^m \to \mathbb{R}^m,$$

Let us interconnect them. How to do this? $f(\,\cdot\,) \rightsquigarrow \tilde{f}(\,\cdot\,,\cdot\,)$

Let $\dot{x} = \tilde{f}_1(x, u)$ be ISS. Let $\dot{y} = \tilde{f}_2(y, v)$ be ISS.

Concider $\begin{cases} \dot{x} = \tilde{f}_1(x, y) \\ \dot{y} = \tilde{f}_2(x, y) \end{cases}$ Is it stable?

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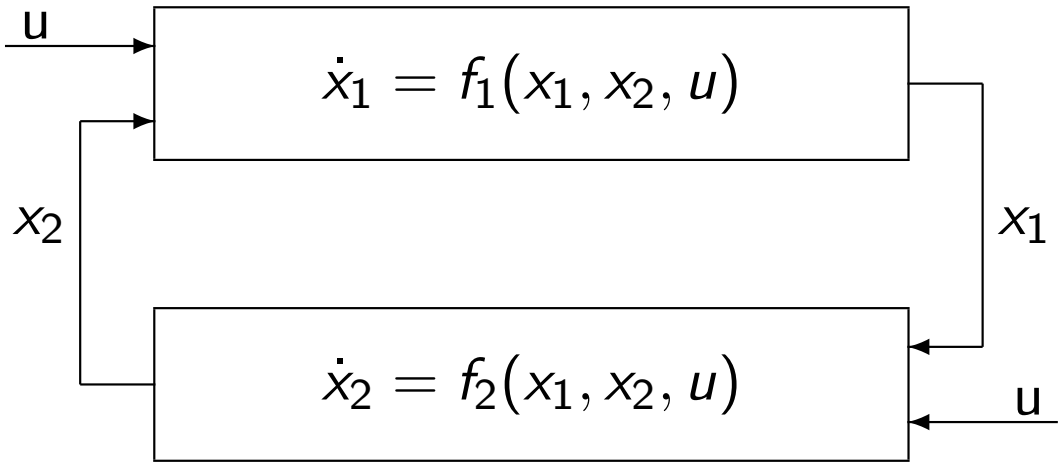
Modelling approach

Two feedback coupled system



Several interconnected systems

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, \ldots, x_n, u) \\ &\vdots \\ \dot{x}_n &= f_n(x_1, \ldots, x_n, u) \end{aligned}$$

Remark: cascades of ISS systems are always ISS

Modelling approach

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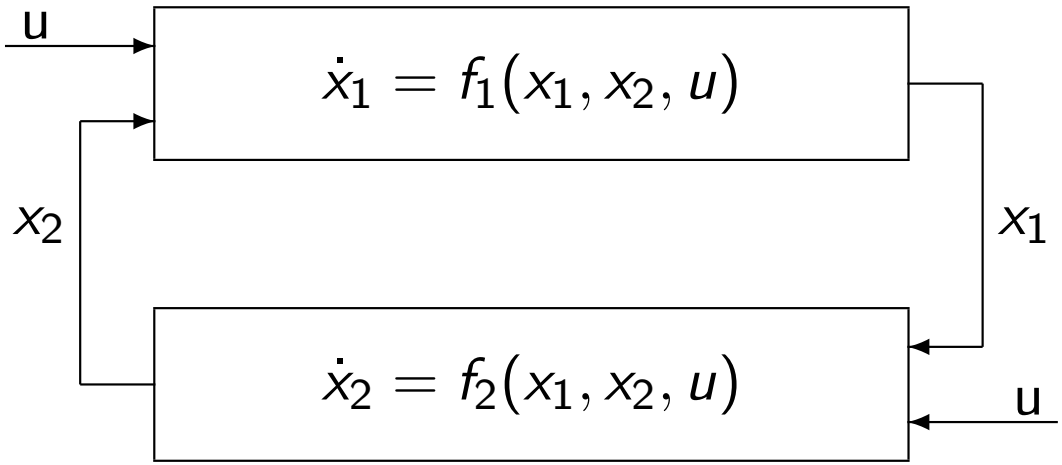

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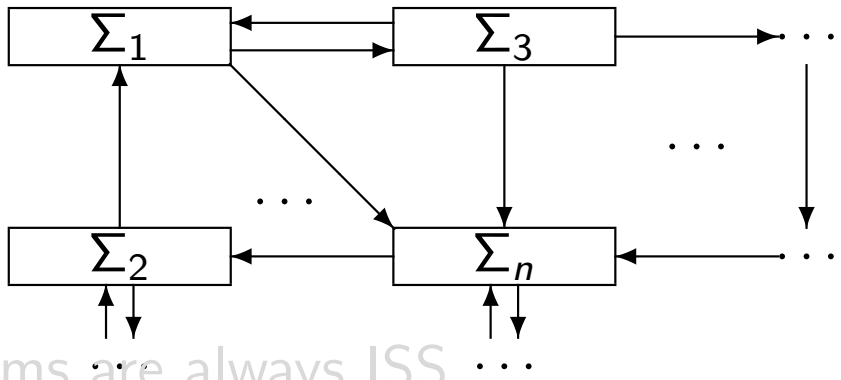

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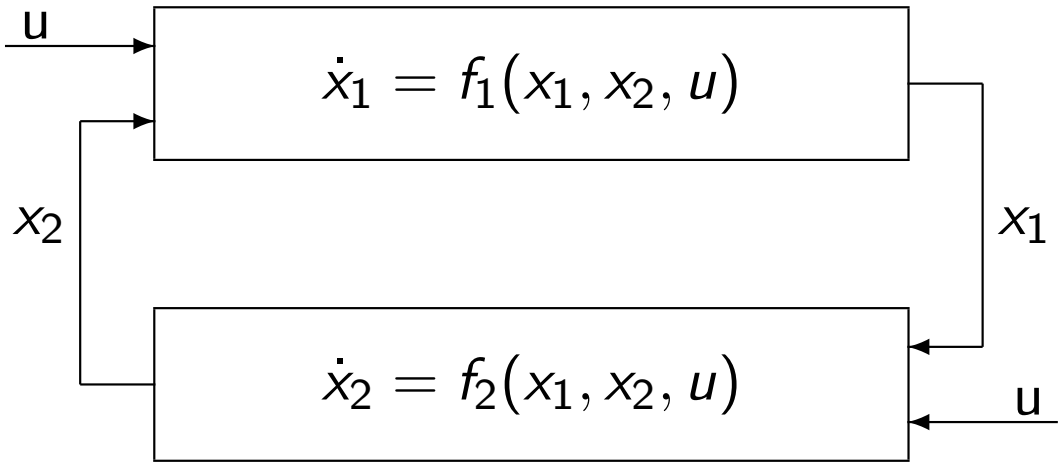

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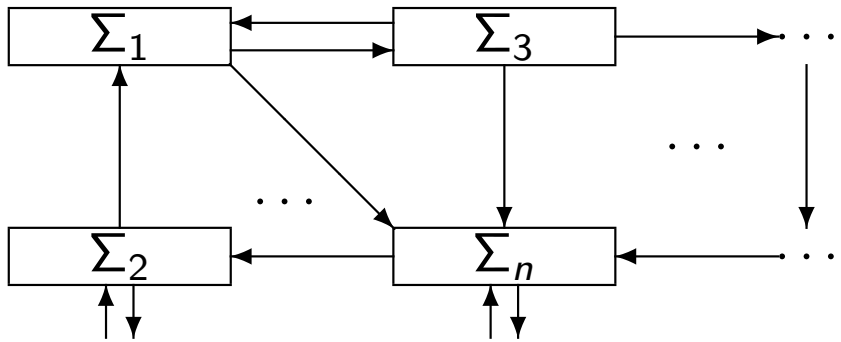

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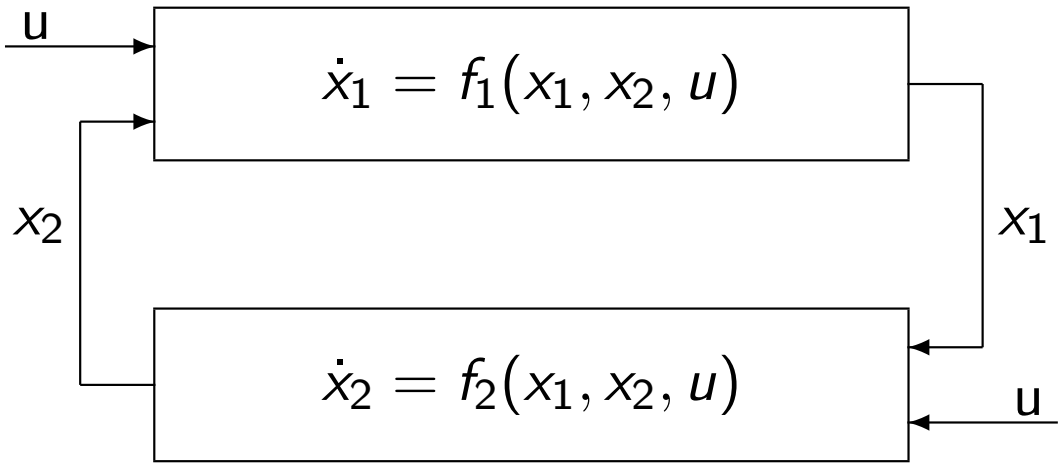

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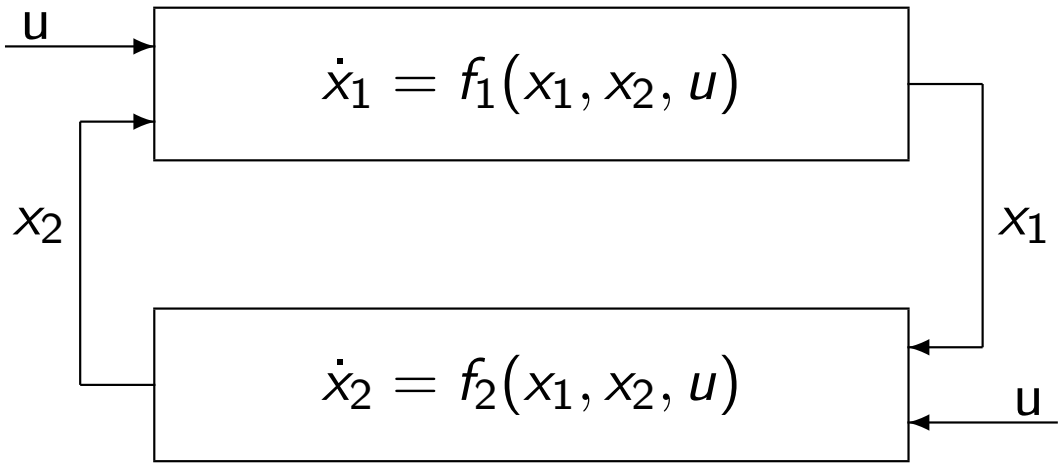

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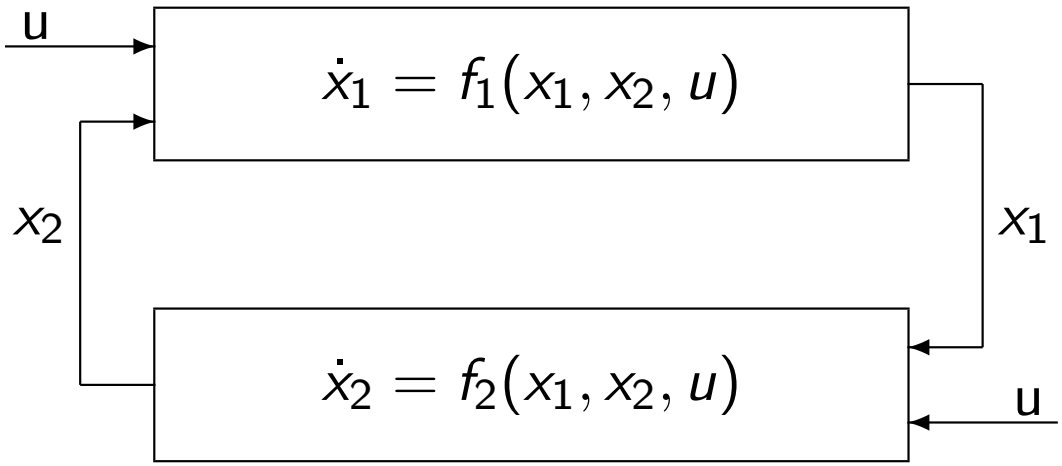

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Lyapunov-Version

$\exists\, \psi_{i1},\, \psi_{i2},\, \chi_1,\, \chi_2 \in \mathcal{K}_\infty$:

$$V_1(x_1) \geq \max\left\{\chi_1(V_2(x_2)),\, \gamma_1(|u|)\right\} \;\Rightarrow\; \nabla V_1(x_1) f_1(x_1, x_2, u) \leq -\alpha_1(V_1)$$

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Theorem 10 (Jiang, Mareels, Wang 1996)

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with ISS-Lyapunov function $V(x_1, x_2) = \max\{\sigma(V_1(x_1)), V_2(x_2)\}$

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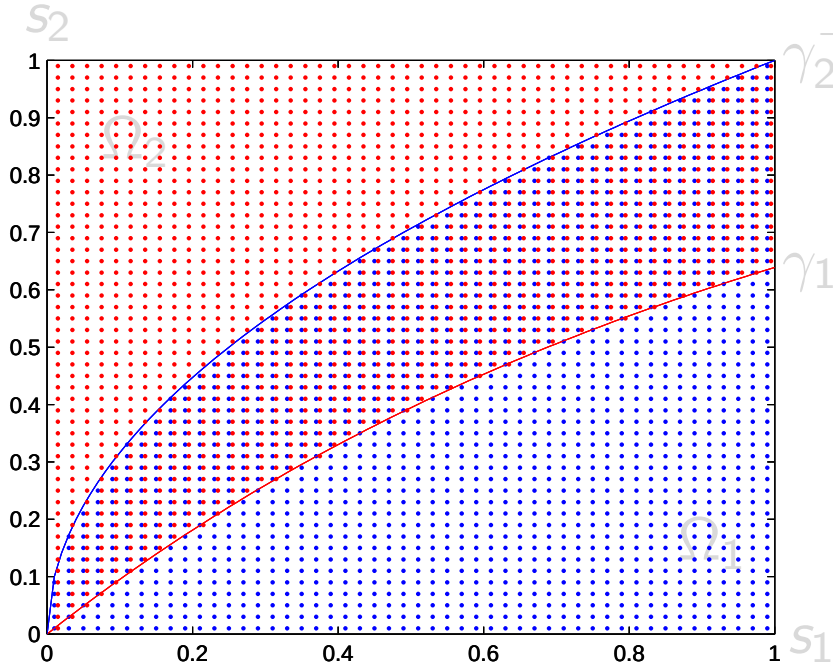

$$\Omega_1: \quad s_1 > \gamma_2(s_2)$$

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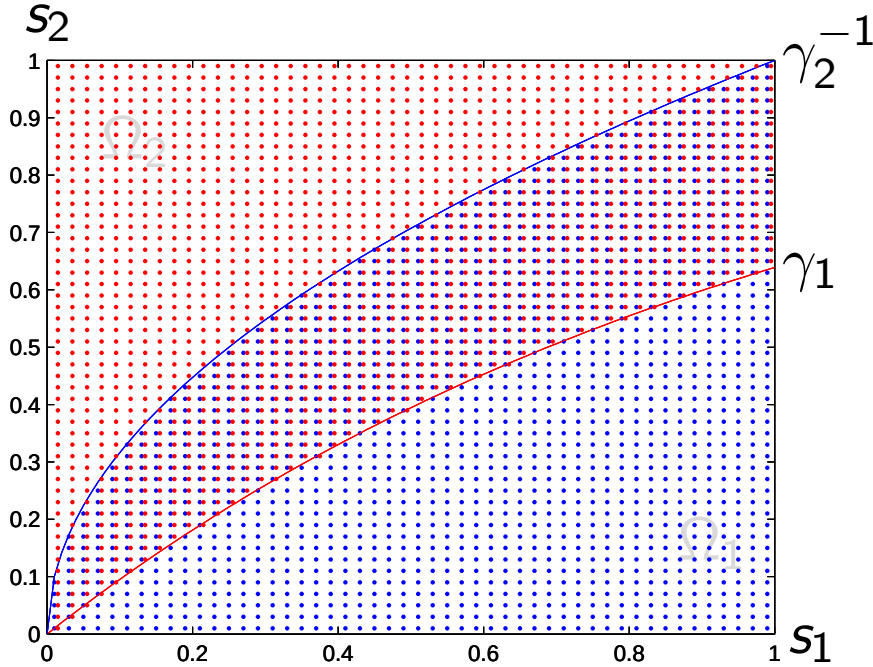

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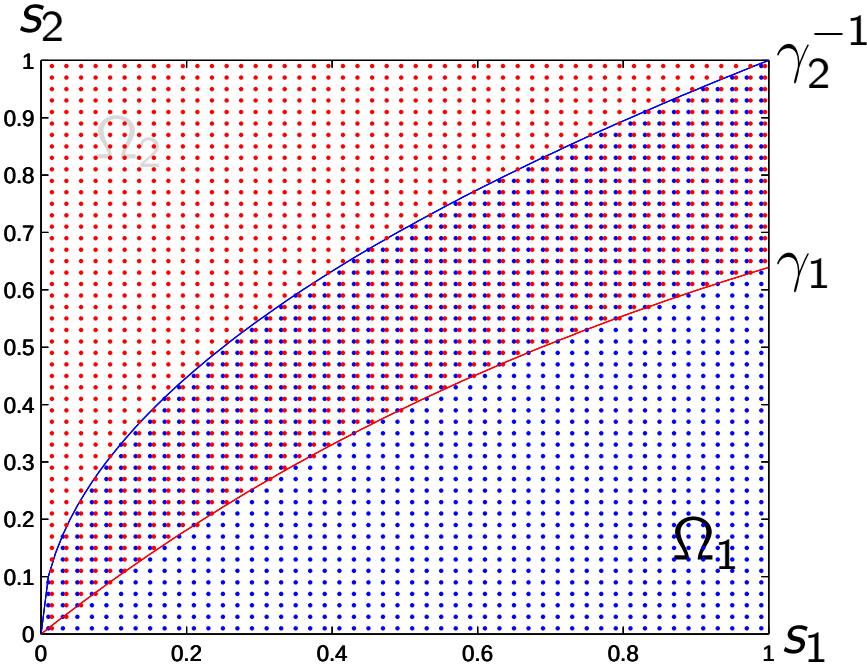

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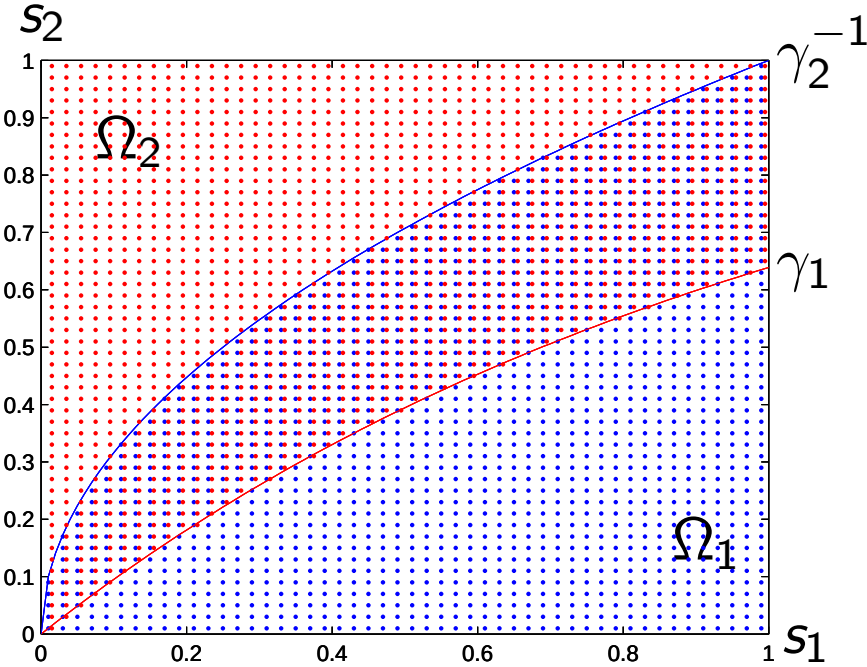

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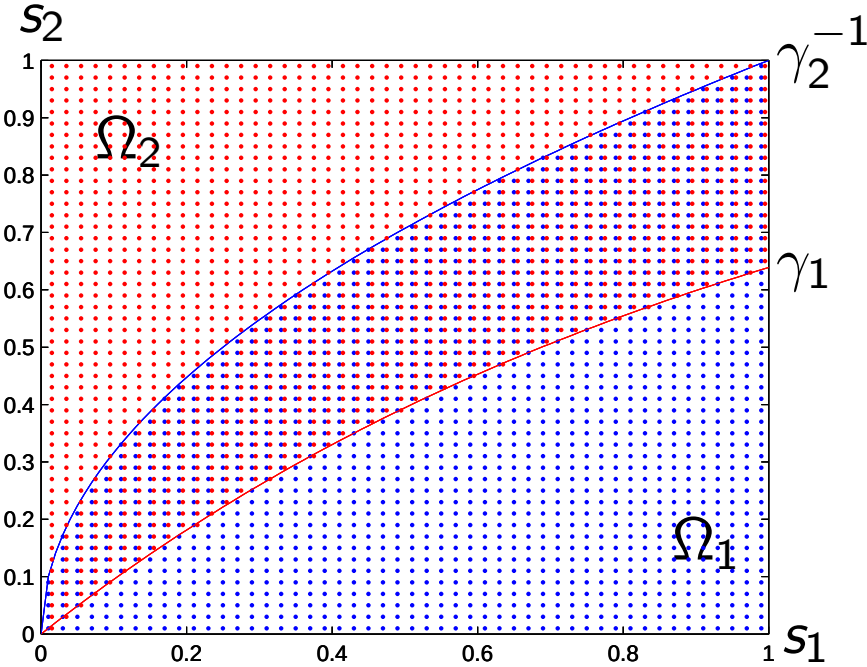


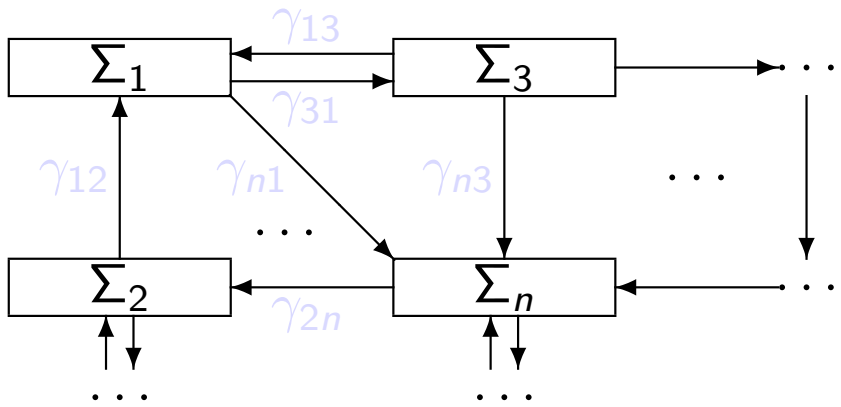
$$\Gamma(s) = \begin{bmatrix} 0 & \gamma_2 \\ \gamma_1 & 0 \end{bmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} \gamma_2(s_2) \\ \gamma_1(s_1) \end{pmatrix} \qquad \Gamma : \mathbb{R}^2_{\geq 0} \to \mathbb{R}^2_{\geq 0}$$

$$\Gamma \not\geqslant \mathrm{Id} \quad \forall\, 0 \neq s = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \in \mathbb{R}^2_{\geq 0} \quad :\Leftrightarrow \quad \exists j \neq i : \ \gamma_i(s_i) < s_j$$

Stability of n coupled systems

Consider

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, \dots, x_n, u) \\ &\vdots \\ \dot{x}_n &= f_n(x_1, \dots, x_n, u) \end{aligned}$$



with

$$|x_i(t)| \leq \max\left\{\beta_i(|x_i(0)|, t), \max_{j=1}^{n} \gamma_{ij}(||x_j||_\infty), \eta(||u||_\infty)\right\}$$

and $\gamma_{ij} \equiv 0$ or $\gamma_{ij} \in \mathcal{K}_\infty$, and $\gamma_{ii} := 0$.

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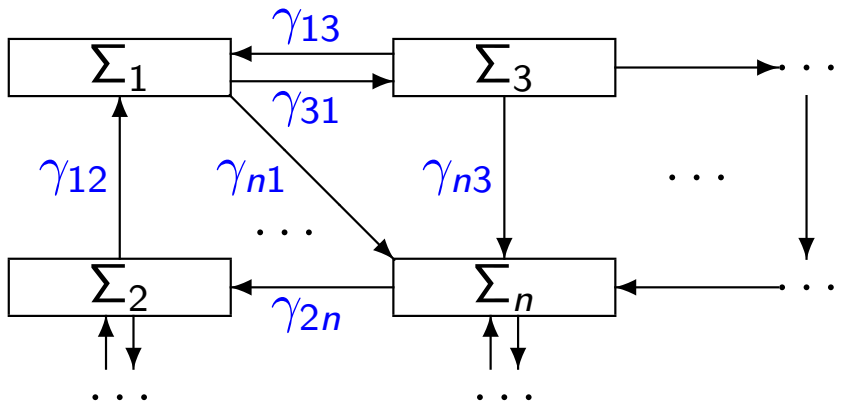

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The gain-Matrix

$$\Gamma : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}, \quad \Gamma(s) = \begin{bmatrix} \max_{j=1}^n \gamma_{1j}(s_j) \\ \vdots \\ \max_{j=1}^n \gamma_{nj}(s_j) \end{bmatrix}$$

Theorem 11 (S.D., B. Rüffer, F. Wirth 2007)

$$\Gamma(s) \not\geq s \quad \forall\, s \in \mathbb{R}^n_{\geq 0},\ s \neq 0 \quad \Rightarrow \quad \dot{x} = f(x,u) \qquad \textit{ISS}$$

Notation: $x = (x_1^T, \ldots, x_n^T)^T$ and $f = (f_1^T, \ldots, f_n^T)^T$

Remark: for $x, y \in \mathbb{R}^n$ ist $x \not\geq y \quad \Leftrightarrow \quad \exists i$ with $x_i < y_i$.

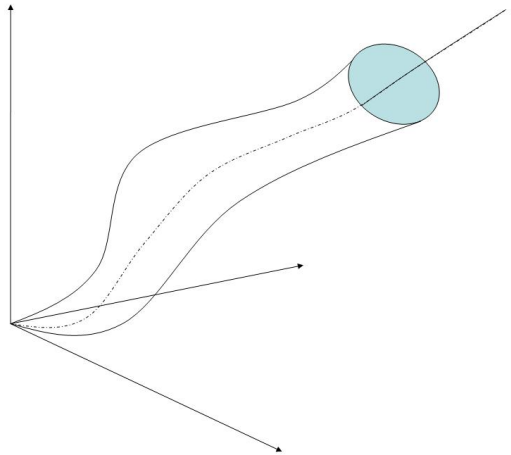
Topological property

Theorem 12 (S.D., B. Rüffer, F. Wirth 2010)

$$\Gamma(s) \not\geq s \quad \forall\, s \in \mathbb{R}^n_{\geq 0} \setminus \{0\} \quad \Rightarrow \quad \exists\, \sigma_1, \ldots, \sigma_n \in \mathcal{K}_\infty :$$

$$\forall\, t > 0 : \quad \Gamma(\sigma(t)) < \sigma(t), \qquad \sigma(t) = (\sigma_1(t), \ldots, \sigma_n(t))^T$$

$$\sigma(t) \in \Omega := \bigcap_{i=1}^{n} \Omega_i \quad \forall t > 0$$



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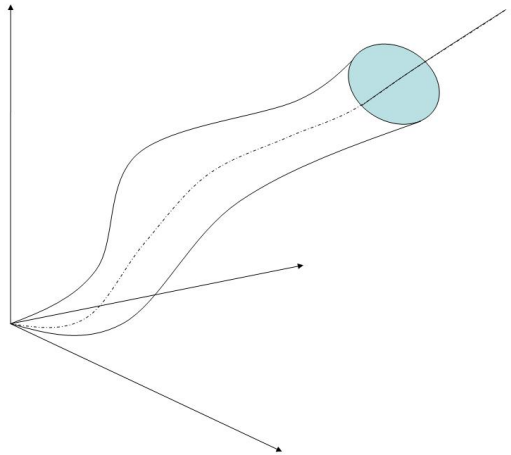

ISS-Lyapunov function for networks

Let V_i be an ISS-Lyapunov function for the i-th system:

$$\psi_{i1}(|x_i|) \leq V_i(x_i) \leq \psi_{i2}(|x_i|), \quad x_i \in \mathbb{R}^{N_i}$$

$$V_i(x_i) \geq \sum_{j=1}^{n} \chi_{ij}(V_j(x_j)) + \gamma_i(|u_i|) \quad \Rightarrow \quad \dot{V}_i(x) \leq -\alpha_i(V_i(x_i))$$

with $\Gamma = (\chi_{ij})_{i,j=1,\dots,n}$, $\Gamma : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$.

Theorem 13 (S.D., B. Rüffer, F. Wirth (2010))

$$\Gamma(s) \not\geq s \quad \forall\, s \in \mathbb{R}^n_{\geq 0} \setminus \{0\} \quad \Rightarrow \quad V(x) = \max_i\{\sigma_i^{-1}(V_i(x_i))\}$$

is ISS-Lyapunov function for $\dot{x} = f(x, u)$.

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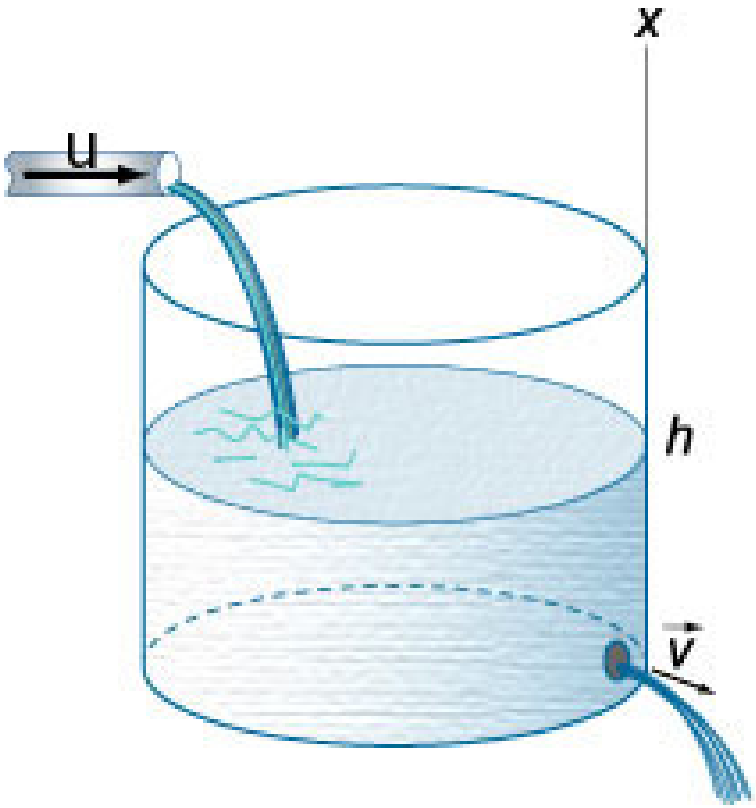
Outline

Concluding remarks



Is $h = h(t)$ bounded
for bounded $u = u(t)$?

Concluding remarks

- ISS provides a robust stability framework for nonlinear systems.
- ISS is a direct extension of the Lyapunov stability for systems with inputs.
- It quantifies the influence of inputs on the systems behavior
- It is popular among engineers for nonlinear control design, observer design, interconnected systems, etc.
- The ISS framework exists for ODEs, PDEs, systems with delays, switched systems, impulsive systems, etc.
- There are almost no ISS works for systems on time scales